JOURNAL OF MATERIALS SCIENCE 15 (1980) 2466–2470

Theoretical aspects of the statistical variation of strength

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A modified version of Weibull's statistical theory of the strength of brittle materials is proposed, in which the expression for failure probability contains an additional term. While this term is negligible when failure originates from a flaw of relatively large size, it becomes increasingly significant as the flaw size is reduced. The resulting revised expressions for failure probability under uniform, uniaxial tension and under Hertzian indentation loading are given, and the effect of a bimodal flaw size distribution is considered in both cases. The implications with regard to the assumed invariance of Weibull statistical parameters under different experimental conditions are discussed.

1. Introduction

The scatter in failure stress data for a brittle material is commonly discussed in terms of the statistical theory of strength due to Weibull [1]. According to this theory, the failure probability can be related to the applied stress by means of an expression involving statistical parameters which are invariant with changes in specimen size or loading configuration. The validity of this approach is re-examined here, in the light of recent advances in the understanding of the relationship between the statistical theory and the actual distribution of flaw sizes in the material.

Hunt and McCartney [2] have derived an expression relating failure probability to the flaw size distribution, and have formally established a relationship with the corresponding statistical expression. However, the present discussion is based on the less generalized analysis presented by Jayatilaka and Trustrum [3], in which it is assumed that the probability of finding a flaw of a given size increases monotonically with decreasing flaw size, and that there are neither stationary points nor points of inflexion in the region of the distribution controlling the experimentally measured strengths. Provided that these conditions are fulfilled the assumption of invariant statistical parameters should be valid.

Not all materials can be expected to have such "well behaved" flaw size distributions [4, 5], and differences in the form of distribution may account for the somewhat contradictory conclusions arising from indentation tests on various glass surfaces. Oh and Finnie [6, 7], for example, found that the observed variation of the mean failure stress with indenter radius for a borosilicate glass was consistent with the predictions of the Weibull theory. A similar conclusion was reached by Hamilton and Rawson [8] from their experiments with soda-lime glasses. In contrast, other workers have reported that the range of failure stresses for soda-lime glasses tended to be increasingly narrower with decreasing indenter radius than anticipated on the basis of either the experimentally derived flaw size distribution [9] or the statistical theory [10]. These discrepancies may be viewed as being equivalent to a variation in the statistical parameters under different loading conditions.

The practical usefulness of the statistical approach is obviously undermined if the parameters which define the failure probability can no longer be considered invariant. In general, this depends on the range of flaw sizes being sampled in a particular experiment. From the practical point of view it would be preferable to make an appropriate correction to the expression for failure probability, while still retaining the advan-

 0022–2461/80/102466–05 $02.50/0

tage of invariant parameters. A suitable modification of the original theory is discussed in the present paper, and this analysis is extended to include the case of a bimodal flaw size distribution.

2. Theoretical discussion

The expression given by Weibull [1] for the failure probability F under uniaxial tension may be generalized as

$$F = 1 - \exp\left[-\int\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \mathrm{d}V\right], \quad (1)$$

where σ is the stress in the volume element $\mathrm{d}V$, σ_u is a threshold stress, σ_0 is a scaling factor and m is the Weibull modulus. If the threshold stress is taken as zero and a uniform stress is applied, this expression reduces to

$$F = 1 - \exp\left[-V\left(\frac{\sigma}{\sigma_0}\right)^m\right], \quad (2)$$

where V is the total volume of the specimen.

Jayatilaka and Trustrum [3] have succeeded in directly relating the mathematical form of the statistical theory to the assumed form of the flaw size distribution. From experimental evidence [11, 12] they concluded that, for an idealized sharp half-crack of length c, the probability density $f(c)$ may be represented by

$$f(c) = \frac{c_0^{n-1}}{(n-2)!}\, c^{-n}\, \mathrm{e}^{-c_0/c}, \quad (3)$$

where c_0 and n are the constants defined in Fig. 1. By integrating this expression over all crack orientations and lengths for which a strain energy density failure criterion [13, 14] is satisfied, they showed that the probability $\mathscr{F}(\sigma)$ of a single crack propagating under a uniform applied stress σ can be expressed as

$$\mathscr{F}(\sigma) = \frac{1}{n!}\left(\frac{\sigma^2 \pi c_0}{K_{\mathrm{IC}}^2}\right)^{n-1}\left[1 - \left(\frac{n-1}{n+1}\right)\frac{\sigma^2 \pi c_0}{K_{\mathrm{IC}}^2} + \ldots\right], \quad (4)$$

where K_{IC} is the mode I critical stress intensity factor. Then, approximating the total failure probability F for a specimen with N cracks by

$$F \simeq 1 - \exp\left[-N\mathscr{F}(\sigma)\right] \quad (5)$$

for large N, and neglecting the second and subsequent terms of the infinite series in Equation 4, they obtained

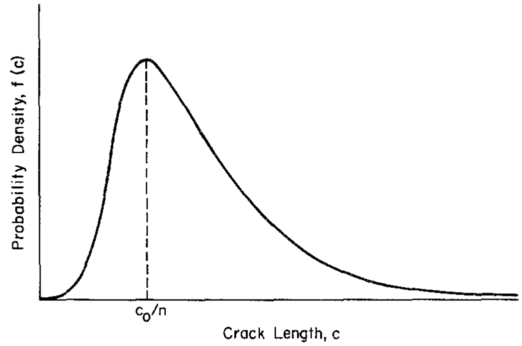

Figure 1 The form of flaw size distribution postulated for a brittle material.

$$F \simeq 1 - \exp\left[-\frac{N}{n!}\left(\frac{\sigma^2 \pi c_0}{K_{\mathrm{IC}}^2}\right)^{n-1}\right] \quad (6)$$

for large N and $\sigma(\pi c_0)^{1/2}/K_{\mathrm{IC}} \ll 1$. Comparison of this expression with Equation 2 reveals that

$$m = 2n - 2. \quad (7)$$

The analysis summarized above requires that the strength controlling flaws be relatively large and lie in the low probability tail of the distribution shown in Fig. 1. When this is not the case the higher order terms in Equation 4 should also be taken into consideration. This is done here approximately by including the second term of the series in addition to the first. The way in which the probability of propagating a single crack actually varies with applied stress is shown in Fig. 2, together with the respective approximations obtained by considering the first term alone and the first two terms together. Since $\mathscr{F}(\sigma)$ cannot sensibly exceed unity the second of these approximations is generally the more physically reasonable, although an upper limit

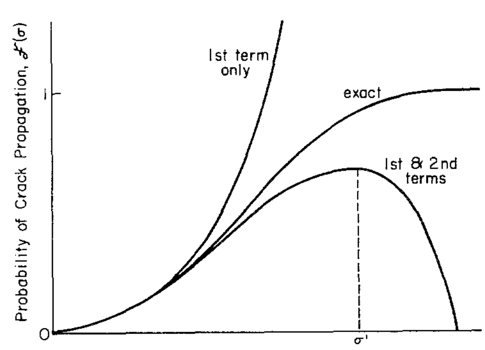

Figure 2 Comparison of the actual probability of propagating a single crack with the approximations discussed in the text.

on the range of its usefulness is imposed as a result of the maximum which occurs at the stress σ', given by

$$\sigma' = \left(\frac{n+1}{n\pi c_0}\right)^{1/2} K_{IC}. \quad (8)$$

Using the new approximation for $\mathscr{F}(\sigma)$, the expression for the failure probability becomes

$$F \simeq 1 - \exp\left\{-\frac{N}{n!}\left[1 - \left(\frac{n-1}{n+1}\right)\frac{\sigma^2\pi c_0}{K_{IC}^2}\right]\left(\frac{\sigma^2\pi c_0}{K_{IC}^2}\right)^{n-1}\right\}, \quad (9)$$

which on rewriting in terms of the statistical notation gives

$$F \simeq 1 - \exp\left\{-V\left[1 - \frac{m}{(m+4)}\left(\frac{\sigma^2\pi c_0}{K_{IC}^2}\right)\right]\left(\frac{\sigma}{\sigma_0}\right)^m\right\}. \quad (10)$$

Suppose now that the flaw sizes are distributed, as in Fig. 3, with a bimodal probability density which takes the form

$$\begin{aligned} f(c) &= f_1(c) + f_2(c) \\ &= \frac{P_1 c_1^{n_1-1}}{(n_1-2)!} c^{-n_1} e^{-c_1/c} \\ &\quad + \frac{P_2 c_2^{n_2-1}}{(n_2-2)!} c^{-n_2} e^{-c_2/c}, \end{aligned} \quad (11)$$

where $P_1 + P_2 = 1$. Provided that $\sigma(\pi c_2)^{1/2}/K_{IC} \ll 1$, which should generally be true in practice, the failure probability for a specimen subjected to a uniform uniaxial tensile stress can be written as

$$F \simeq 1 - \exp\left\{-V\left[\left(1 - \frac{m_1}{(m_1+4)}\left(\frac{\sigma^2\pi c_1}{K_{IC}^2}\right)\right) \times \left(\frac{\sigma}{\sigma_1}\right)^{m_1} + \left(\frac{\sigma}{\sigma_2}\right)^{m_2}\right]\right\}, \quad (12)$$

where m_1, σ_1 and m_2, σ_2 represent the values

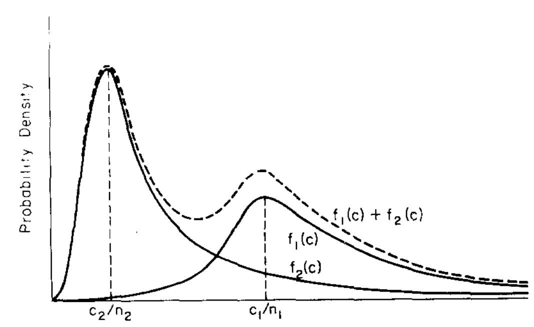

Figure 3 The form of a bimodal flaw size distribution.

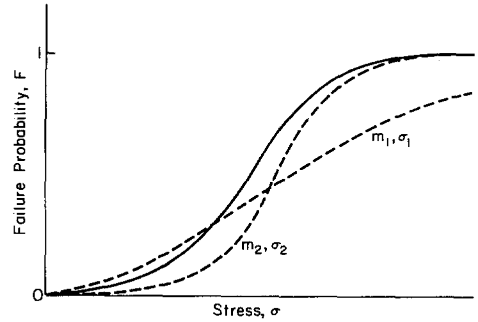

Figure 4 The modified theory for a bimodal flaw size distribution (solid line) in comparison with the original theory applied to the component distributions (broken lines).

towards which the effective statistical parameters tend for very large and very small specimens, respectively. A comparison of the form of the resulting failure probability distribution for specimens of intermediate size, with those expected from the original theory (Equation 2) using the individual pairs of parameters, is made in Fig. 4.

3. Applications to indentation strength

The stress field around a spherical indenter in contact with an isotropic elastic half-space has been treated by Huber [15] as a special case of the theory of contact between elastic bodies orginally developed by Hertz [16]. For an indenter of radius R with an applied load P the contact circle has a radius a, given by

$$a = \left[\tfrac{3}{4}PR\left(\frac{1-\nu_i^2}{E_i} + \frac{1-\nu_s^2}{E_s}\right)\right]^{1/3}, \quad (13)$$

where ν_i and ν_s are the Poisson's ratios, and E_i and E_s are the Young's moduli of the indenter and specimen, respectively. At the periphery of the contact circle the radial tensile stress in the surface attains its maximum value σ_a, which is related to the contact radius by

$$\sigma_a = \frac{2(1-\nu_i)a}{3\pi\left(\frac{1-\nu_i^2}{E_i} + \frac{1-\nu_s^2}{E_s}\right)R} = \frac{a}{K}. \quad (14)$$

Outside the contact circle, the radial tensile stress σ_r at a radius r from the centre is given by

$$\sigma_r = \sigma_a\left(\frac{a^2}{r^2}\right). \quad (15)$$

Because the stress within the specimen becomes compressive at a small distance below the surface, the indentation test effectively samples only those flaws lying in the surface itself. Consequently, the volume integral in Equation 1 reduces to one taken over the surface only, and the probability of failure can therefore be expressed as

$$F = 1 - \exp\left[-\int_a^\infty \left(\frac{\sigma_r}{\sigma_0}\right)^m 2\pi r \, dr - \int_0^a \left(\frac{r}{K\sigma_0}\right)^m 2\pi r \, dr\right], \quad (16)$$

in which the second integral represents the risk of failure in that area now in compression beneath the indenter during the time it was initially in tension [7]. After integration and slight rearrangement, this gives

$$F = 1 - \exp\left[-\pi a^2 \left(\frac{3m}{(m-1)(m+2)}\right)\left(\frac{\sigma_a}{\sigma_0}\right)^m\right], \quad (17)$$

where $m \neq 1$. If a correction involving the second term in Equation 4 is made as previously, the expression for failure probability becomes

$$F \simeq 1 - \exp\left\{-\int_a^\infty \left[1 - \frac{m}{(m+4)}\left(\frac{\sigma^2 \pi c_0}{K_{IC}^2}\right)\right] \times \left(\frac{\sigma_r}{\sigma_0}\right)^m 2\pi r \, dr - \int_0^a \left[1 - \frac{m}{(m+4)} \times \left(\frac{\sigma^2 \pi c_0}{K_{IC}^2}\right)\right]\left(\frac{r}{K\sigma_0}\right)^m 2\pi r \, dr\right\}. \quad (18)$$

From this can be readily obtained

$$F \simeq 1 - \exp\left\{-\pi a^2 \left[\frac{3m}{(m-1)(m+2)} - \frac{3m(m+2)}{(m+1)(m+4)}\left(\frac{\sigma^2 \pi c_0}{K_{IC}^2}\right)\right]\left(\frac{\sigma_a}{\sigma_0}\right)^m\right\}, \quad (19)$$

where $m \neq 1$. For a bimodal distribution of flaw sizes the resulting modified expression for failure probability is

$$F \simeq 1 - \exp\left\{-\pi a^2 \left[\left(\frac{3m_1}{(m_1-1)(m_1+2)} - \frac{3m_1(m_1+2)}{(m_1+1)(m_1+4)}\left(\frac{\sigma^2 \pi c_1}{K_{IC}^2}\right)\right)\left(\frac{\sigma_a}{\sigma_1}\right)^{m_1} + \frac{3m_2}{(m_2-1)(m_2+2)}\left(\frac{\sigma_a}{\sigma_2}\right)^{m_2}\right]\right\}, \quad (20)$$

where $\sigma(\pi c_2)^{1/2}/K_{IC} \ll 1$ and $m_1, m_2 \neq 1$.

4. Discussion

The analysis presented here provides theoretical insight into the inability of the Weibull theory in some cases to reconcile failure stress data obtained with radically different specimen sizes or loading configurations. Improving the approximation for the relationship between failure probability and the flaw size distribution by including an additional term of an infinite series expansion has allowed the development of a modified expression for failure probability containing a stress dependent correction term. While this term is negligible for relatively low stress failure due to a large flaw in the low probability tail of the distribution, so that the expression for failure probability reduces to that given by Weibull, it becomes increasingly significant as the failure stress increases (i.e., as the flaw size decreases). Without such a correction the use of parameters determined from relatively large flaw sizes will lead to increasing over-estimation of the failure probability as the experimentally sampled range tends towards smaller flaws.

If the flaw size distribution is bimodal in form, the failure probability can be more accurately represented in terms of two sets of statistical parameters, each of which corresponds to one of the individual flaw populations which comprise the aggregate distribution. Variation of the loading conditions alters the range of flaw sizes being sampled experimentally, and thus affects the relative contributions of the constituent flaw size distributions to the total failure probability. In general, the failure probability will be under-estimated by the unmodified Weibull theory if parameters corresponding to a higher flaw size range than that being presently sampled are employed, due to the increased contribution of the high density of relatively small flaws in the second park of the distribution. Since, as Ernsberger [5] has pointed out, at least two discrete populations of flaws may be expected to exist in typical glass surfaces, the modified theory developed here is clearly of practical significance.

Loading conditions may be conveniently altered by carrying out Hertzian tests with indenters of various radii. This technique offers the potential for exploring a relatively wide portion of the flaw size distribution, without the inherent difficulties associated with specimen edge effects which are encountered in performing flexural tests on brittle materials. Such considerations have made the Hertzian test popular experimentally, in particular for examining the validity of the Weibull theory. The application of the modified statistical theory to this type of loading is therefore of special interest.

Finally, it must be emphasized that it is essential to distinguish clearly between the familiar statistical size effect and the phenomena deriving from the present modified analysis. The original theory recognizes that the probability of finding a flaw of a given size depends on specimen dimensions. However, it considers neither the effect of the range of the flaw size distribution sampled experimentally, nor that of the form of the distribution itself.

Acknowledgement

The author thanks N. H. Macmillan for his useful comments.

References

1. W. WEIBULL, *Ingeriörsvetenskapsakad. Handl.* **151** (1939) 45.
2. R. A. HUNT and L. N. McCARTNEY, *Int. J. Fracture* **15** (1979) 365.
3. A. DE S. JAYATILAKA and K. TRUSTRUM, *J. Mater. Sci.* **12** (1977) 1426.
4. A. S. ARGON, *Proc. Roy. Soc.* **A250** (1959) 482.
5. F. M. ERNSBERGER, "Progress in Ceramic Science" Vol. 3, edited by J. E. Burke (Pergamon Press, London, 1963) p. 58.
6. H. L. OH and I. FINNIE, *J. Mech. Phys. Solids* **15** (1967) 401.
7. *Idem, Int. J. Fracture Mech.* **6** (1970) 287.
8. B. HAMILTON and H. RAWSON, *J. Mech. Phys. Solids* **18** (1970) 127.
9. Y. M. TSAI and H. KOLSKY, *ibid.* **15** (1967) 29.
10. D. G. RICKERBY, Ph.D. Dissertation, Cambridge University (1977).
11. J. D. POLONIECKI and T. R. WILSHAW, *Nature* **229** (1971) 226.
12. T. R. WILSHAW, *J. Phys. D* **4** (1971) 1567.
13. G. C. SIH, *Int. J. Fracture* **10** (1974) 305.
14. A. DE S. JAYATILAKA, I. J. JENKINS and S. V. PRASAD, Proceedings of the 4th International Conference on Fracture (University of Waterloo, Ontario, Canada, 1977).
15. M. T. HUBER, *Ann. Phys.* **14** (1904) 153.
16. H. HERTZ, "Miscellaneous Papers" (Macmillan, London, 1896).

Received 8 February and accepted 3 March 1980.